KINETICS OF THE DEVELOPMENT OF AN INTERCRYSTALLITE CRACK BY MEANS OF DIFFUSION MASS TRANSFER

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1. The study of fracture at high temperature is of both theoretical and practical interest. At high temperatures and low stresses fracture develops by the propagation of a crack along the grain boundaries [1]. These cracks are called intercrystallite cracks.

The development of cracks along the grain boundaries is due to a number of physical factors. First of all, splitting of the grains along the boundaries at high temperatures is capable of initiating cracks. These cracks grow perferentially along the boundaries due to a weakening of the cohesive forces (a reduction in the surface energy of the fracture) at the boundary, and also due to accelerated diffusion of atoms along the boundaries. The main time up to fracture is usually taken up in the development of the cracks [1].

In this paper we consider the problem of the kinetics of the development of an intercrystallite crack due to diffusion mass transfer. While the crack increases, its volume increases due to diffusion transfer of material from the tip of the crack along the grain boundary. This mechanism by which the crack grows has been observed in experiments on a number of brittle materials [2]. A similar problem was formulated and solved previously by numerical methods [2]. Below, we present an analytical solution for the stationary case.

2. We will consider the following model of a stationarily growing crack shown in Fig. 1. The crack occupies the half-plane $\xi < 0$, $\eta = 0$ in a moving system of coordinates connected with the tip of the crack (at the initial instant of time t = 0 the axes of the fixed system of coordinates x, y and the moving system of coordinates ξ, η coincide). For a stationarily increasing crack we have

$$\xi = x - vt,\ \eta = y,$$

where v is the rate of growth of the crack (the crack grows from left to right along the x axis). All the quantities in the stationary mode can be represented by functions of ξ, η or x − vt, y. The external loads produce at the tip of the crack a stress intensity factor K, so that along the continuation of the plane of the crack when $\xi > 0$ normal slipping stresses $\sigma_{yy}(\xi) = \sigma_0(\xi)$ act, where

$$\sigma_0(\xi) = \frac{K}{\sqrt{2\pi(\xi + R)}}, \tag{2.1}$$

and R is the radius of curvature of the tip of the crack. In this formulation the accurate shape of the tip of the crack is not investigated. Expression (2.1) takes into account approximately the finiteness of the crack thickness, and as $R \to 0$ reduces to the usual equation for the stresses from a crack-slit [1]. The introduction of $R \neq 0$, as will be seen below, is essential for the concrete formulation of the problem. When normal stresses $\chi(x, t)$ are applied the chemical potential $\sigma(x, t)$ of the atoms at the boundary decreases by an amount [2, 3]

$$\chi(x, t) = -\Omega\sigma(x, t),$$

where Ω is the atomic volume. Due to the action of the gradient $\chi(x, t)$ a flow of atoms $J(x, t)$ occurs in a thin layer of thickness δ along the boundary

$$J(x, t) = -\frac{D\delta}{\Omega T}\frac{\partial\chi(x, t)}{\partial x} = \frac{D\delta}{T}\frac{\partial\sigma(x, t)}{\partial x},$$

where D is the self-diffusion coefficient along the boundary, δ is the effective thickness of the boundary, and T is the temperature (in ergs). If we assume that the width of the

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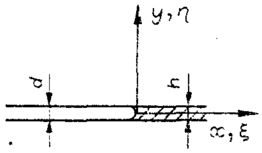

Fig. 1

crack d remains constant, the velocity of the crack v is given by

$$v = \frac{\Omega}{d} J(x, t)\big|_{x=vt} = \frac{\varkappa}{d\mu}\frac{\partial\sigma(x, t)}{\partial x}\bigg|_{x=vt}. \tag{2.2}$$

where μ is the shear modulus, and $\varkappa = D\delta\Omega\mu/T$. The actual stresses at the boundary $\sigma(x, t)$ are made up of the stresses (2.1) from the friction and the stresses $\sigma_1(x, t)$ due to the layer of deposited material (the material transferred from the tip of the crack and deposited ahead of it along the boundary, forming a layer of thickness h(x, t))

$$\sigma(x, t) = \sigma_0(x, t) + \sigma_1(x, t). \tag{2.3}$$

The stresses $\sigma_1(x, t)$ can be calculated as the total stresses from the distributed dislocations (for more detail see [1])

$$\sigma_1(x, t) = \int\limits_{vt}^{\infty} \rho(x', t)\frac{\mu}{2\pi(1-\nu)}\left[\frac{1}{x-x'}\sqrt{\frac{x'-vt}{x-vt}}\right]dx', \tag{2.4}$$

where $\rho(x, t) = -\partial h(x, t)/\partial x$ is the dislocation distribution density, and ν is Poisson's ratio. The material balance equation takes the form

$$\frac{\partial h(x, t)}{\partial t} = -\Omega\frac{\partial J(x, t)}{\partial x} = \frac{\varkappa}{\mu}\frac{\partial^2\sigma(x, t)}{\partial x^2}$$

or, taking into account $\partial/\partial t = -v d/d\xi$, $\partial/\partial x = d/d\xi$, $dh/d\xi = -\rho(\xi)$

$$-\frac{\varkappa}{v\mu}\frac{d^2\sigma(\xi)}{d\xi^2} = \rho(\xi). \tag{2.5}$$

Substituting (2.5) into (2.4) and the result obtained into (2.3), we obtain for the total normal stresses $\sigma(\xi)$ at the boundary

$$\sigma(\xi) = \sigma_0(\xi) - \beta\int\limits_0^{\infty}\sigma''(\xi')\left[\frac{1}{\xi-\xi'}\sqrt{\frac{\xi'}{\xi}}\right]d\xi', \tag{2.6}$$

where $\sigma''(\xi)$ is the second derivative with respect to ξ, and $\beta = \varkappa/2\pi(1-\nu)v$. Equation (2.6) for $\sigma(\xi)$ must be solved with additional boundary conditions. The first condition (2.2) when ξ = 0 can be written in the form

$$\sigma'(0) = vd\mu/\varkappa. \tag{2.7}$$

Further, the chemical potential of the atoms when ξ = 0 at the boundary and at the tip of the crack should be the same, which is equivalent to the condition

$$\sigma(0) = 2\gamma/d, \tag{2.8}$$

where γ is the surface energy density of the crack. Moreover, the stresses at infinity should vanish

$$\sigma(\infty) = 0. \tag{2.9}$$

Integrating (2.6) with respect ξ we obtain the normalization condition for

$$\int\limits_0^{\infty}[\sigma(\xi) - \sigma_0(\xi)]\,d\xi = 0. \tag{2.10}$$

Equation (2.6) in its initial form does not enable one to use well-known analytical methods of solution. Hence, we will carry out some preliminary conversion of the equation and the boundary condition. Putting $\varphi(\xi) = \sigma''(\xi)\sqrt{\xi}$, Eq. (2.6) can be written in the form

$$\int_0^\infty \varphi(\xi') \frac{d\xi'}{\xi' - \xi} = \frac{\sqrt{\xi}}{\beta}\,[\sigma(\xi) - \sigma_0(\xi)]. \tag{2.11}$$

Equation (2.11) is an equation with a Cauchy-type kernel [4], a solution of which, bounded at $\xi = 0$ and $\xi = \infty$, exists in view of (2.10) and is given by the expression

$$\varphi(\xi) = \sigma''(\xi)\sqrt{\xi} = -\frac{\sqrt{\xi}}{\beta\pi^2}\int_0^\infty [\sigma(\xi') - \sigma_0(\xi')]\frac{d\xi'}{\xi' - \xi}.$$

Introducing the new desired function $u(\zeta) = \sigma(\zeta) - \sigma_0(\zeta)$, and the dimensionless variable $\zeta = \xi/\pi\sqrt{\beta}$, we obtain

$$u''(\zeta) + \int_0^\infty u(\zeta')\frac{d\zeta'}{\zeta' - \zeta} = f(\zeta), \tag{2.12}$$

where

$$f(\zeta) = -\frac{3}{8\pi\sqrt{\beta}}\frac{K}{\left(\zeta + \dfrac{R}{\pi\sqrt{\beta}}\right)^{5/2}}.$$

The additional conditions for $u(\zeta)$, taking (2.1) and (2.7)-(2.9) into account, can be written in the form

$$u(0) = u_1 \equiv (2\gamma/d - K/\sqrt{2\pi R}); \tag{2.13}$$

$$u(\infty) = 0; \tag{2.14}$$

$$u'(0) = u_2 \equiv \pi\sqrt{\beta}(vd\mu/\varkappa + K/2R\sqrt{2\pi R}). \tag{2.15}$$

3. We will now consider the solution of the problem. Equation (2.12) is satisfied on the positive semi-axis ζ. We will extend the equation to the whole of the ζ axis by assuming the functions $u(\zeta)$ and $f(\zeta)$ to be zero when $\zeta < 0$. As a result we obtain

$$u''(\zeta) + \int_{-\infty}^\infty u(\zeta')\frac{d\zeta'}{\zeta' - \zeta} = f(\zeta) + V(\zeta), \tag{3.1}$$

where

$$V(\zeta) = \begin{cases} 0, & \zeta > 0, \\ \displaystyle\int_0^\infty u(\zeta')\frac{d\zeta'}{\zeta' - \zeta}, & \zeta < 0. \end{cases}$$

Applying a Fourier transformation to (3.1) $f(\lambda) = \int_{-\infty}^\infty f(\zeta)e^{i\lambda\zeta}d\zeta$, we obtain a Wiener–Hopf equation [5] for $u(\lambda)$:

$$(-\lambda^2 + \pi i \operatorname{sign}\lambda)u(\lambda) = f(\lambda) + V(\lambda) + u_2 - i\lambda u_1. \tag{3.2}$$

Here the additional terms on the right side occur due to the integration of $\int_{-\infty}^\infty u''(\zeta)e^{i\lambda\zeta}d\zeta$ by parts taking into account the boundary conditions (2.13)-(2.15). We will denote by $G(\lambda) = (-\lambda^2 + \pi i \operatorname{sign}\lambda)$ the symbol of the operator and represent it in the form

$$G(\lambda) = -(\lambda^2 + 1)\left(1 - \frac{\pi i \operatorname{sign}\lambda + 1}{\lambda^2 + 1}\right) \equiv -(\lambda^2 + 1)\,g(\lambda).$$

The main step in the solution of (3.2) is the factorization of $G(\lambda)$, i.e., the representation of $G(\lambda)$ in the form of the product

$$G(\lambda) = G_+(\lambda)G_-(\lambda),$$

where $G_+(\lambda)$, $G_-(\lambda)$ are functions holomorphic and different from zero in the upper and lower half plane respectively. Using the C-theorem (see [5]), we obtain in this case

$$G_+(\lambda) = (\lambda + i)\exp[\Gamma_+(\lambda)] \equiv (\lambda + i)g_+(\lambda),$$
$$G_-(\lambda) = -(\lambda - i)\exp[-\Gamma_-(\lambda)] \equiv -(\lambda - i)g_-(\lambda),$$

where

$$\Gamma_\pm(\lambda) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln[1 - (\pi i \operatorname{sign}\omega + 1)(\omega^2 + 1)^{-1}]}{\lambda \pm i0 - \omega}\,d\omega.$$

We divide both sides of (3.2) by $G_-(\lambda)$

$$G_+(\lambda)u(\lambda) = G_-^{-1}(\lambda)f(\lambda) + G_-^{-1}(\lambda)V(\lambda) + u_2G_-^{-1}(\lambda) - i\lambda u_1G_-^{-1}(\lambda). \tag{3.3}$$

Here $G_+(\lambda)u(\lambda)$ is a function that is analytic in the upper half plane, while the last three terms on the right side of (3.3) are analytic in the lower half plane. The mixed term $G_-^{-1}(\lambda)f(\lambda)$ can be expanded in a sum of functions that are analytic in the appropriate half planes

$$G_-^{-1}f(\lambda) = F_+(\lambda) + F_-(\lambda),$$

where

$$F_\pm(\lambda) = \pm\frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{G_-^{-1}(\omega)f(\omega)}{\lambda \pm i0 - \omega}\,d\omega.$$

Separating in (3.3) terms with different analytical regions we obtain

$$G_+(\lambda)u(\lambda) - F_+(\lambda) = G_-^{-1}(\lambda)V(\lambda) + F_-(\lambda) + u_2G_-^{-1}(\lambda) - i\lambda u_1G_-^{-1}(\lambda).$$

Hence, using Liouville's theorem we have

$$G_+(\lambda)u(\lambda) - F_+(\lambda) = C = \text{const}. \tag{3.4}$$

From (3.4) we obtain

$$u(\lambda) = CG_+^{-1}(\lambda) + \frac{iG_+^{-1}(\lambda)}{2\pi}\int_{-\infty}^{\infty} \frac{G_-^{-1}(\omega)f(\omega)}{\lambda + i0 - \omega}\,d\omega. \tag{3.5}$$

The behavior of $u(\lambda)$ as $\lambda \to \infty$ is related to the behavior of $u(\zeta)$ as $\zeta \to 0$, so that taking (2.13) and (2.15) into account we have in the neighborhood of infinity

$$u(\lambda) = \frac{i}{\lambda}u_1 - \frac{1}{\lambda^2}u_2 - O\left(\frac{1}{\lambda^3}\right). \tag{3.6}$$

On the other hand, expanding the solution (3.5) in powers of λ^{-1} we have

$$u(\lambda) = C\lambda^{-1} + \left[C\left(-i + \frac{i}{2\pi}\int_{-\infty}^{\infty}\ln g(\lambda)\,d\lambda\right) + \frac{i}{2\pi}\int_{-\infty}^{\infty} G_-^{-1}(\omega)f(\omega)\,d\omega\right]\lambda^{-2} + O(\lambda^{-3}). \tag{3.7}$$

Comparing (3.6) and (3.7) we find the value of the unknown constant $C = iu_1$ and obtain the following relation between u_1 and u_2:

$$u_2 = u_1\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\ln g(\lambda)\,d\lambda - 1\right) - \frac{i}{2\pi}\int_{-\infty}^{\infty} G_-^{-1}(\lambda)f(\lambda)\,d\lambda. \tag{3.8}$$

Equation (3.8), taking into account expressions (2.13) and (2.15) for u_1 and u_2, is an implicit equation defining the dependence of the stationary rate of development of the crack v on the stress intensity factor K: $v = v(K)$. In general, Eq. (3.8) can only be solved numerically. For large crack velocities one can use the asymptotic expansion (3.8) in powers of v. Omitting the intermediate calculations we will merely give the final expression for $v = v(K)$ for large crack velocities (we assume $2R = d$)

$$v = \frac{1-\nu}{\pi^2}\left(\frac{C_1}{2\pi} - 1\right)^2 \frac{\varkappa K^2}{\mu^2 d^3}, \tag{3.9}$$

where $C_1 \sim 1$ is a constant defined by the integral

$$C_1 = \int_{-\infty}^{\infty} \ln \frac{\omega^4 + \pi^2}{(\omega^2 + 1)^2}\, d\omega.$$

Considering Eq. (3.8) as $v \to 0$, we see that the stationary development of the crack is only possible at velocities greater than a certain critical value v_{min}

$$v \geqslant v_{\min} \simeq \gamma\varkappa/\mu d^3.$$

Experimental investigations give $v \sim K^n$, where n varies over a wide range. Thus, in Nikonel-718 alloy for large crack velocities n = 1-2.5 [2], which is close to the theoretical value n = 2 in (3.9).

LITERATURE CITED

1. H. Liebowitz (editor), Fracture [Russian translation], Vol. 1, Mir, Moscow (1973).
2. V. Vitek, "A theory of diffusion controlled intergranular creep crack growth," Acta Metall., 26, No. 9 (1978).
3. Sh. Kh. Khannanov, "Statics and kinetics of vacancies in a nonuniformly stressed crystal," Fiz. Met. Metalloved., 45, No. 3 (1978).
4. F. D. Gakhov, Boundary Value Problems [in Russian], Fizmatgiz, Moscow (1963).
5. B. Noble, The Use of the Wiener—Hopf Method to Solve Differential Equations in Partial Derivations [Russian translation], IL, Moscow (1962).